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DepH – Mathematics

TOPIC – Convergence of Infinite positive series (Necessary Condition of Convergence, p-series Test – Comparison Test

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BY – Dr. Shree Nath Sharma

(Associate prof.)

Necessary condition for convergence of a positive Term series

A Positive Term series $\sum u_n$ is convergent if $\lim_{n\to\infty} u_n = 0$

Proof

Let $\sum_{n=1}^{\infty} u_n$ be a series of positive Terms

Let $S_n = u_1 + u_2 + u_3 + \cdots + u_n$

$S_{n-1} = u_1 + u_2 + u_3 + \cdots + u_{n-1}$

$\therefore S_n - S_{n-1} = u_n$

Now this series is convergent if

$\lim_{n\to\infty} S_n$ = Finite and Unique.

Therefore, Let $\lim_{n\to\infty} S_n = S$ = a unique Finite quantity

Therefore $\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1})$

$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$

$= S - S$

$= 0$

Thus we see that Necessary Condition for convergent of a positive infinite series $\sum_{n=1}^{\infty} u_n = 0$

But this condition is not sufficient

Remarks :→ If $u_n \to 0$, as $n \to \infty$

we are not sure whether the series $\sum u_n$ is convergent or not but if $\lim_{n\to\infty} u_n \neq 0$ then series $\sum u_n$

Ex:→ Test the series

$$\sqrt{\frac{1}{2}} + \sqrt{2/3} + \sqrt{\frac{3}{4}} + \sqrt{4/5} + \cdots\cdots$$

Solution :→ Here Given Series is

$$\sqrt{1/2} + \sqrt{2/3} + \sqrt{3/4} + \sqrt{4/5} + \cdots$$

Since the Given Series is of positive Terms

Now $u_n = \sqrt{\frac{n}{n+1}}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n}{n(1+1/n)}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{1+1/n}} = 1 \neq 0$$

this this series of positive Terms is not convergent.

Ex:→ Test the series $\sum_{n=1}^{\infty} \frac{1}{(1+1/n)^n}$

Solution :→ Given series is

$$\sum_{n=1}^{\infty} \frac{1}{(1+1/n)^n}$$

Here All the Terms of the given Series is +ve

Also $u_n = \frac{1}{(1+1/n)^n}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$

$$= \frac{1}{e} \quad \text{Since } \lim_{n\to\infty}\left(1+\frac{1}{x}\right)^{x} =$$

$\neq 0$

thus this series of positive terms is divergent.

* Hyperharmonic or p-series Test:

Statement :→ The infinite series $\sum \frac{1}{n^p}$ is always divergent except when p is positive and $p > 1$

Solution :→ The infinite series $\sum \frac{1}{n^p}$ is called Hyperharmonic Series

Here $\sum u_n = \sum \frac{1}{n^p}$

the following cases arise

$p = 1$, $p < 1$ or $p > 1$

se I
when $p=1$

then $\sum u_n = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

$\sum u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (A) (AS $p=1$)

since $3 < 4$

$\Rightarrow \frac{1}{3} > \frac{1}{4}$

$\Rightarrow \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

thus $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$

similarly $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$

$\Rightarrow \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{2}$ and so on

similarly $\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{16} > \frac{8}{16} = \frac{1}{2}$

$\therefore 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots$

$\cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

$+ \cdots$

$+ \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

or $\sum u_n > \frac{1}{\cdot} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

the series on right hand side is divergent as the sum of 1st n terms of the series

$\lim_{n \to \infty} S_n = \infty$

Case II →

When $p<1$ or Negative

In this case $p<1$ so $n^p < n \;\forall n \in N$

$$\therefore \frac{1}{n^p} > \frac{1}{n} \quad \forall n \in N$$

thus in this case

$$\sum \frac{1}{n^p} > \sum \frac{1}{n} \text{ ——— (11)}$$

But the series $\sum \frac{1}{n}$ is diverg.

Hence from (11) we find th

In this case the Given series $\sum u_n$

is also divergent.

Case III,

When $p>1$ and positive.

Since $3^p > 2^p$

$$\Rightarrow \frac{1}{3^p} < \frac{1}{2^p}$$

$$\Rightarrow \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p}$$

$$\Rightarrow \frac{1}{2^p} + \frac{1}{3^p} < \frac{2}{2^p}$$

Similarly $\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{4}{4^p}$

Similarly

$$\frac{1}{8^p} + \frac{1}{9^p} + \cdots + \frac{1}{15^p} < \frac{8}{8^p}$$ an

on

The given series

$$\sum u_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots\cdots$$

$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \cdots + \frac{1}{8^p}\right) + \cdots$$

$$\sum u_n < \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots\cdots$$

The series on Right hand side is a geometric progression whose common ratio is $\frac{2}{2^p}$ or $\frac{1}{2^{p-1}}$ which is less than 1 as $p > 1$ and +ve

Hence this geometric series is convergent and we find that the sum of the given series is less than this geometric series and hence

$\sum_{n=1}^{\infty} u_n$ is convergent.

Comparison Test For convergence of infinite positive series

Statement :→ If Σu_n and Σv_n be two Given series and if $\lim_{n \to \infty} \frac{u_n}{v_n}$ = Finite and non zero then Σu_n and Σv_n are either both convergent or both divergent.

Proof :→ let $\Sigma u_n = u_1 + u_2 + u_3 + \cdots$

and $\Sigma v_n = v_1 + v_2 + v_3 + \cdots$

be two Given infinite Series

As $\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3} \ldots$ are all Finite,

Let $\frac{u_3}{v_3}$ have the Greatest value

u amongst the Fractions $\frac{u_1}{v_1}, \frac{u_2}{v_2}$ etc

Then $\frac{u_1}{v_1} < u \Rightarrow u_1 < u v_1$

$\frac{u_2}{v_2} < u \Rightarrow u_2 < u v_3$

$\frac{u_3}{v_3} = u \Rightarrow u_3 = u v_3$

$\frac{u_4}{v_4} < u \Rightarrow u_4 < u v_4$ and so on

Adding these results we have

$u_1 + u_2 + u_3 + u_4 + \cdots < v_1 u + v_2 u + v_3 u + v_4 u + \cdots$

$\Rightarrow u_1 + u_2 + u_3 + u_4 + \cdots < u(v_1 + v_2 + v_3 + v_4 + \cdots)$

$$\Rightarrow \frac{u_1 + u_2 + u_3 + u_4 \cdots}{v_1 + v_2 + v_3 + v_4 + \cdots} < u$$

$$\Rightarrow \frac{\Sigma u_n}{\Sigma v_n} < u \text{ ——— (I)}$$

In a similar way if $\frac{u_2}{v_2}$ has the least value u' amongst the fractions

$\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}, \frac{u_4}{v_4} \ldots$ etc

then $\frac{u_1}{v_1} > u' \Rightarrow u_1 > v_1 u'$

$\frac{u_2}{v_2} = u' \Rightarrow u_2 = v_2 u'$

$\frac{u_3}{v_3} > u' \Rightarrow u_3 > u' v_3$ and so on.

Adding these results we have

$$\frac{\Sigma u_n}{\Sigma v_n} > u' \text{ ——— (II)}$$

From (I) and (II)

$$u' < \frac{\Sigma u_n}{\Sigma v_n} < u$$

i.e the value of

$\frac{\Sigma u_n}{\Sigma v_n}$ lies between the

Least value u' and the Greatest value u of the fractions

$\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}, \ldots$ and

each of these ~~Nonzero~~ Nonzero and Finite

$\Rightarrow \frac{\Sigma u_n}{\Sigma v_n} = \lambda$ (say) a Finite quantity — (iii)

Now if the series Σv_n is convergent then its sum is unique and finite and so from (iii) we find sum of the series Σu_n is also finite and unique, λ being a finite quantity.

$\Rightarrow$ If the Σv_n is convergent then Σu_n is also convergent.

Similarly if Σv_n is divergent then its sum $\rightarrow \infty$ or $(-\infty)$

and λ being finite from (iii)
we find that the sum of the series Σu_n
also tends to $+\infty$ or $(-\infty)$

$\Rightarrow$ the series Σv_n is divergent then
the series Σu_n is also divergent.

<u>Working Rule of Comparison Test</u> —

We can compare the given series Σu_n
with auxiliary ~~series~~ series Σv_n.
The convergency or divergency of the auxiliary
series should already be known to us.
and then if we can prove

$\lim_{n\to\infty} \frac{u_n}{v_n}$ is finite, then the
series Σu_n and Σv_n converge or diverge
to-gether.

<u>Method for Finding the Auxilary Series</u>

<u>Step I</u> $\rightarrow$ Find u_n (nth term of the given series)

If this u_n contains power of n only
then retain the term containing the highest
power of n only and denote them by v_n
and thus we have the auxiliary series Σv_n